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On Some Nonlinear Spaces of Approximating Functions

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1. INTRODUCTION

Given f in $C[a, b]$, the space of real-valued continuous functions on the interval $[a, b]$, the best approximation in the uniform norm by an element of π_n, the polynomials degree $\leqslant n$, is attained by a polynomial whose properties are governed by the Tchebycheff theory (cf. [2]). If a better approximation is desired, then one may enlarge the approximating space to π_{n+1}, a linear space of dimension $n+2$. In this paper we consider an enlargement of π_n to a space designated as V_n. That V_n might be a useful space for approximation purposes was suggested to us by Norman H. Painter. Let F be a function which is continuous and strictly positive on $(-\infty, \infty)$. For a real number α, let $V_n(\alpha)$ be the space of functions of the form $F(\alpha x) P(x)$, where P belongs to π_n. Let V_n be the union (over all real α) of the $V_n(\alpha)$. The situation may be generalized to consider general Haar systems, rather than just π_n. While some of our results will be valid in that context, we shall be content to state and prove our results for the special Haar system π_n.

Although V_n is not a linear space, it is the union of linear spaces, $V_n(\alpha)$, each of which has the Haar property. Moreover, $V_n(0) = \pi_n$, and the underlying parameter space of V_n has dimension $n+2$. Hence, as a space of approximating functions, it could be compared reasonably with π_{n+1}. In this regard V_n has both advantages and disadvantages. Naturally, if f, the function to be approximated, is "F-like," then V_n is preferable. More concretely, consider the case $F(x) = e^x$. Then a function ϕ ($\not\equiv 0$) of V_n may have up to n zeros in $[a, b]$, and this is true also of all its derivatives. A function of π_{n+1} may have up to $n+1$ zeros, but its derivatives ($\not\equiv 0$) will have less. Thus if f has several changes of curvature, an approximating function from V_n may be preferable to one from π_{n+1}. On the other hand, any f can be interpolated by a function of π_{n+1} at any $n+2$ points. Such

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interpolation by a function of V_n is not possible if the sign changes of F at the points are not consistent with the fact that a function of V_n has at most n zeros. Thus, if f has several sign changes, approximation by a function of π_{n+1} may be preferable. As noted above, each $V_n(\alpha)$ is a linear space with the Haar property. Hence the Tchebycheff theory applies to the best approximation over $V_n(\alpha)$. If a best approximation ϕ exists in V_n, then it occurs in some space $V_n(\alpha)$. Hence $f-\phi$ has at least $n+2$ points in an alternant.

As a function space, V_n has several desirable properties. For any choice of F, it is dilation invariant; i.e., if ϕ in in V_n, then so is $\phi(cx)$ for any scalar c. In the case that $F(x)=e^x$, V_n is also translation invariant. For the same F, differentiation of V_n functions produce V_n functions, as it does for π_{n+1}. But for $\alpha \neq 0$, differentiation reproduces each space $V_n(\alpha)$. Again with $F(x)=e^x$, V_n has "property Z" (cf. [3, p. 3]); i.e., two different functions of V_n may intersect in at most m points, where m depends only on n. In fact, this V_n bears some resemblance to the example of Rice on exponentials (cf. [3, p. 42]). In this theory, property Z is a key ingredient in proving existence of a best approximation. In our own treatment (cf. Theorem 2), we make use of a growth condition and this allows consideration of many spaces V_n, where property Z is not available.

Section 2 is devoted to existence theorems: i.e., for F with specified properties, every f of $C[a,b]$ has a best approximation in V_n. That this must be proved is shown by the examples of Section 4, where, for convenience, all examples have been collected.

In the third section, the uniqueness of the best approximation is discussed. The subject being difficult and not fitting into known theories, our results are sparse. However, something can be said in a particular case.

The existence of a best approximation from class V_n for every f in $C[a,b]$ depends primarily on the behaviour of F at ∞ and at $-\infty$. Thus it is convenient to treat separately the cases $\alpha \geqslant 0$ and $\alpha \leqslant 0$ and to consider only intervals $[a,b]$ for which $0 \leqslant a < b < \infty$. In the matter of notation, let

$$E(\alpha, n, F, f, a, b) = E(\alpha) = \inf_{P \text{ in } \pi_n} \|f(x) - F(\alpha x)\, P(x)\|.$$

Where there is no possibility of confusion, the simpler notation, $E(\alpha)$, will be used. By the ordinary Tchebycheff theory, the polynomial P giving the best approximation is uniquely defined, and there is an alternant of at least $n+2$ points. Let

$$E^{(+)} = \inf_{\alpha \geqslant 0} E(\alpha), \qquad E^{(-)} = \inf_{\alpha \leqslant 0} E(\alpha), \qquad E = \min(E^{(+)}, E^{(-)}).$$

Since

$$\inf_{\alpha \text{ real},\, P \text{ in } \pi_n} \|f(x) - F(\alpha x)\, P(x)\| = \inf_{\alpha \text{ real}} \inf_{P \text{ in } \pi_n} \|f(x) - F(\alpha x)\, P(x)\|,$$

the number E measures the deviation from the best approximation. It is a useful and easily verified fact that the function $E(\alpha)$ is continuous in α. In view of this, the nonexistence of a best approximation can occur only if

$$\liminf_{\alpha \to \infty} E(\alpha) = E \qquad \text{or} \qquad \liminf_{\alpha \to -\infty} E(\alpha) = E.$$

2. Existence Theorems

The hypotheses of our three existence theorems all involve the existence of a limit for F, finite or not, at ∞. The conclusions involve the existence of an $\alpha \geqslant 0$ and a P in π_n for which $E^{(+)}$ is attained. The results can then be applied, where appropriate, to $F(-x)$ in place of $F(x)$ to obtain the existence of $\alpha \leqslant 0$ and P in π_n for which $E^{(-)}$ is attained.

Our first theorem is the simplest to state and to prove. That it cannot be substantially improved is shown by Examples 4.1–4.3.

THEOREM 1. *Let* $\lim_{x \to \infty} F(x) = A$, *which is finite and positive. For any* f *in* $C[a, b]$, *there is an* $\alpha \geqslant 0$ *for which* $E(\alpha) = E^{(+)}$.

Denote by $Q_\alpha(x) = F(\alpha x) P_\alpha(x)$ that unique function of $V_n(\alpha)$ such that $\|f - Q_\alpha\| = E(\alpha)$. Since the function which is identically 0 belongs to each $V_n(\alpha)$, then $E(\alpha) \leqslant \|f\|$ and $\|Q_\alpha\| \leqslant 2\|f\|$ for each α. Under the present circumstances, there exists an $M > 0$ such that $0 < 1/F(\alpha x) \leqslant M$ for all x in $[a, b]$ and all $\alpha \geqslant 0$. Thus, for all $\alpha \geqslant 0$,

$$\|P_\alpha\| \leqslant 2M\|f\|. \tag{1}$$

Let

$$\liminf_{\alpha \to \infty} E(\alpha) = E(\infty). \tag{2}$$

Let $\{\alpha_m\}$ denote a sequence tending to ∞ such that lim inf in (2) becomes lim. By virtue of (1), there exists a subsequence, also denoted by α_m, such that P_{α_m} converges uniformly in $[a, b]$ to P, a polynomial of π_n. If $a > 0$, $F(\alpha_m x)$ converges uniformly on $[a, b]$ to A. If $a = 0$, we take $\varepsilon > 0$. Then $F(\alpha_m x)$ converges uniformly on $[a + \varepsilon, b]$ to A. Thus Q_{α_m} converges uniformly here to $AP(x)$, a function of $V_n(0)$. Hence

$$\sup_{a \in \varepsilon \leqslant x \leqslant b} |f(x) - AP(x)| \leqslant E(\infty).$$

We may now take the limit of the left side, a monotone function of ε, which increases as ε decreases to 0. Since $f-AP$ is continuous, this limit is $\|f-AP\|$. Thus $E(0) \leqslant \|f-AP\| \leqslant E(\infty)$. Thus a best approximation must be attained by some Q_α, $0 \leqslant \alpha < \infty$.

The point of our next theorem is that existence will hold when $\lim_{x\to\infty} F(x)$ is 0 or ∞ if the convergence is rapid enough. Taken together, the two parts of the theorem imply, for example, that if $F(x)=e^x$, then, for each continuous f, there exists α such that $E(\alpha)=E$.

THEOREM 2. *Let F be continuous and strictly positive on $[0, \infty)$. Assume that* (i) *for each $x > 1$.*

$$\lim_{\beta\to\infty} \frac{F(\beta x)}{F(\beta)} = 0,$$

or (ii) *for each $x > 1$, this same limit is ∞. Then, for each f in $C[a, b]$, there exists α in $[0, \infty)$ such that $E(\alpha) = E^{(+)}$.*

Hypothesis (ii) combined with the assumption of monotonicity of F yields a function which is "order positive" in the terminology of Roulier and Varga (cf. [4], where the concept is used for different purposes).

It is enough to consider the situation for which the first hypothesis (i) is satisfied. The modifications in the proof for hypothesis (ii) are minor. It is also enough to assume that the interval $[a, b]$ is $[0, 1]$. In fact, the proof is somewhat simpler if $a > 0$.

Let $E(0, 0)$ denote the best approximation to f by constant functions,

$$E(0,0) = \inf_c \|f-c\|.$$

If, for every α, $E(\alpha) \geqslant E(0,0)$, there is no problem. The best approximation is given by a constant function. Thus, let is assume that there exists $\varepsilon > 0$ such that for some values of α, $E(\alpha) \leqslant (1-2\varepsilon)\, E(0,0)$. Let A_ε denote the set of such α,

$$A_\varepsilon = \{\alpha \geqslant 0 \mid E(\alpha) \leqslant (1-2\varepsilon)\, E(0,0)\}.$$

It suffices to show that the nonempty set A_ε is bounded, hence compact. As before, write $Q_\alpha(x) = F(\alpha x)\, P_\alpha(x)$. We have

$$f(x) - E(\alpha) \leqslant Q_\alpha(x) \leqslant f(x) + E(\alpha), \qquad 0 \leqslant x \leqslant 1. \tag{3}$$

Since f may be approximated by the zero function, $E(0,0) \leqslant \|f\|$. We may assume that $\|f\| = f(\bar{x})$ for some $\bar{x}$ in $[0, 1]$. Thus, there exists a subinterval,

$J=[a,b]$ of $[0,1]$ with $0<a$, such that $f(x)\geqslant(1-\varepsilon)E(0,0)$ for all x in J. Hence, for x in J and α in A_ε,

$$f(x)-E(\alpha)\geqslant[f(x)-(1-\varepsilon)E(0,0)]+\varepsilon E(0,0)\geqslant\varepsilon E(0,0)>0.$$

Combining this inequality with (3) gives the existence of positive constants M and N such that,

$$\frac{M}{F(\alpha x)}\leqslant P_\alpha(x)\leqslant\frac{N}{F(\alpha x)},\qquad x \text{ in } J,\ \alpha \text{ in } A_\varepsilon, \tag{4}$$

where P_α is determined by its values on any $n+1$ points $x_0, x_1,\dots, x_n$ in $[0,1]$. We shall choose the points in the interval $J=[a,b]$ so that $x_0=a$, $x_n=(a+b)/2$, and the other points are equally spaced between them. Write

$$\frac{Q_\alpha(x_i)}{F(\alpha x_i)}=P_\alpha(x_i)=\sum_{j=0}^{n}\gamma_j x_i^j,$$

where

$$\gamma_j=\sum_{i=0}^{n}\frac{A_{ij}}{F(\alpha x_i)}=\frac{1}{F(\alpha x_n)}\sum_{i=0}^{n}\frac{F(\alpha x_n)}{F(\alpha x_i)}A_{ij}.$$

The numbers A_{ij} depend only on the points x_i and the values $Q_\alpha(x_i)$. Thus there are bounds for A_{ij} independent of α. Furthermore $F(\alpha x_n)/F(\alpha x_i)\leqslant 1$ for sufficiently large α by virtue of hypothesis (i). Hence

$$|\gamma_j|\leqslant\frac{C}{F(\alpha x_n)},\qquad j=0,1,\dots,n;$$

C independent of α. It follows from (4) that

$$\frac{M}{F(\alpha b)}\leqslant P_\alpha(b)\leqslant\sum_{j=0}^{n}|\gamma_j|\,b^j\leqslant\frac{C'}{F(\alpha x_n)}$$

or

$$0<M\leqslant C'F(\alpha b)/F(\alpha x_n).$$

The limit, as $\alpha\to\infty$, is 0. If A_ε were unbounded, we would have $M=0$, contradicting (4). Hence, as desired, the set A_ε is bounded.

Our last existence theorem concerns the situation when F is a rational function which is positive on $(-\infty,\infty)$. Thus its convergence to ∞ or to 0 at ∞ is slow relative to the functions considered in Theorem 2. Example 4.2 involves a rational F for which the degree of the numerator exceeds that of the denominator. The nonexistence in Example 4.2 is typical of this situation. In Example 4.3, nonexistence follows with another rational F for $C[a, b]$, where $a > 0$. These examples explain and justify the statement of Theorem 3. We remark that when the degrees are equal, the result follows directly from Theorem 1. Our proof depends heavily on using the special Haar system π_n.

THEOREM 3. *Let F be a rational function, positive on $(-\infty, \infty)$ and such that the degree of the numerator does not exceed that of the denominator. For every f in $C[0, 1]$, there exists α in $[0, \infty)$ such that $E(\alpha) = E^{(+)}$.*

It may be assumed that $F(x) = S(x)/T(x)$, where both S and T are real positive polynomials on $(-\infty, \infty)$ with degree $S \leqslant$ degree T. Let the leading term of S be $b_M x^M$ and that of T be $c_N x^N$, where $M \leqslant N$. Then

$$\frac{1}{F(\alpha x)} = d(\alpha x)^{N-M} \frac{1 + C(\alpha x)}{1 + B(\alpha x)}, \qquad d = c_N/b_M, \tag{5}$$

where B and C are rational functions such that $B(\alpha x)$ and $C(\alpha x)$ are uniformly small for large α and for x in $[\varepsilon, 1]$, $0 < \varepsilon < 1$. Thus, for x in this same interval, there is a constant C_3 such that,

$$\frac{\alpha^{M-N}}{F(\alpha x)} \leqslant C_3.$$

Since $\|Q_\alpha\| \leqslant 2\|f\|$,

$$|P_\alpha(x)| \leqslant \frac{2\|f\|}{F(\alpha x)} \leqslant 2C_3 \|f\| \alpha^{N-M}, \qquad \varepsilon \leqslant x \leqslant 1.$$

Evaluation of P_α at $n+1$ fixed points in, say, $[\frac{1}{2}, 1]$ with $0 < \varepsilon < \frac{1}{2}$ shows that its coefficients satisfy a similar inequality; there is a constant C_4 such that

$$|a_j(\alpha)| \leqslant C_4 \alpha^{N-M}, \qquad \text{where} \quad P_\alpha(x) = \sum_{j=0}^{n} a_j(\alpha) x^j. \tag{6}$$

Another estimate for the coefficients of P_α is required, and this depends on the fact that $a = 0$ is the left end point of our interval. For x in the interval $[0, 1/\alpha]$, there is a bound for $1/F(\alpha x)$, say C_5, which is independent of α. Since $\|Q_\alpha\| \leqslant 2\|f\|$,

$$|P_\alpha(x)| \leqslant 2C_5 \|f\|, \qquad 0 \leqslant x \leqslant 1/\alpha.$$

Thus $P_\alpha(x/\alpha)$ has this same bound in $[0, 1]$ so that there exists C_6 such that

$$|a_j(\alpha)| \leqslant C_6 \alpha^j, \qquad j = 0, 1,..., n. \tag{7}$$

As above, let

$$\liminf_{\alpha \to \infty} E(\alpha) = E(\infty),$$

and let $\alpha(m)$ be a sequence such that lim inf becomes lim. By (6), each of the $n+1$ sequences $a_j(\alpha(m))[\alpha(m)]^{M-N}$, $j = 0, 1,..., n$, is bounded. Thus, for a subsequence of $\alpha(m)$, also denoted by $\alpha(m)$, there is convergence of each of these $n+1$ sequences. Let

$$\lim_{m \to \infty} a_j(\alpha(m))[\alpha(m)\Big]^{M-n} = a_j.$$

We note that, because of (7), $a_j = 0$ if $j < N - M$. Using (5), we have

$$\begin{aligned} &d a_j(\alpha(m))\, F(\alpha(m)x)\, x^j \\ &\quad = [\alpha(m)]^{M-n} a_j(\alpha(m))\, x^{M-N-j} \left\{ \frac{1 + B(\alpha(m)x)}{1 + C(\alpha(m)x)} \right\}. \end{aligned}$$

For x in $[\varepsilon, 1]$, this converges uniformly to $a_j x^{M-N+j}$ so that $\lim_{m\to\infty} Q_{\alpha(m)}(x) = (1/d) \sum_{j=N-M}^{n} a_j x^{M-N+j} = P(x)$, uniformly in $[\varepsilon, 1]$. Since $M \leqslant N$, P is an element of $\pi_n = V_n(0)$. Because of the choice of $\alpha(m)$,

$$\sup_{\varepsilon \leqslant x \leqslant 1} |f(x) - P(x)| \leqslant E(\infty).$$

Since this is true for every $\varepsilon > 0$,

$$E(0) \leqslant \|f - P\| \leqslant E(\infty).$$

3. Uniqueness

Because of the complicated intersection theory involved, not much of a positive nature about uniqueness can be said. Rather artificial and uninteresting examples can be constructed which do exhibit uniqueness. For example, let $F(x)$ be identically 1 for $x \leqslant 1$ and equal to x for $x \geqslant 1$. Consider f in $C[1, 2]$ and restrict α to $[1, \infty)$; V_n is then the linear span of x, $x^2,..., x^{n+1}$ so that both existence and uniqueness follow.

For a more interesting example, such as $F(x) = e^x$, the situation is simple only when the degree n is 0. Here α is unrestricted, and f is in the space $C[a, b)$ with $0 \leqslant a < b$. Any two different functions from V_0 can intersect in at most one point. By methods related to the "betweeness property" and "zero sign compatibility" (cf. [1]), one can show uniqueness easily.

For $n \geqslant 1$, with the same F, one can show by the usual Rolle's theorem argument that two different functions of V_n may intersect in as many as $2n+1$ points. Since this is no smaller than the dimension of the parameter space, then dificulties may follow (cf. [2, p. 146]). In fact, we may show, by example, that uniqueness does not hold even for $n=1$. For definiteness, take the interval as $[0, 1]$, and let $Tx = 1-x$, a transformation taking $[0, 1]$ onto itself, and such that V_1 is preserved. Thus, if $g(x) = e^{\alpha x}(ax+b)$, $(g \circ T)(x) = e^{-\alpha x}(-ax + (a+b))\, e^{\alpha}$. Except when g is a constant, these are different functions. Let f be a function of $C[0, 1]$ which is invariant under T; i.e., $f(x) = (f \circ T)(x)$ for x in $[0, 1]$. Let g be a best approximation to f from V_1. Since

$$\|f - g\| = \|(f \circ T) - (g \circ T)\| = \|f - (g \circ T)\|,$$

then $g \circ T$ is also a best approximation, and a different one unless g is constant. Such an f is produced in Example 4.4. It is a fact of some interest that for $n=1$, there are no more than two best approximations.

THEOREM 4. *Let $F(x) = e^x$, and $n = 1$. For any f in $C[0, 1]$, there exist at most two different functions of V_1 giving a best approximation.*

The details of the proof are fairly complicated, and we content ourselves with an outline. Let Q_α and Q_β be the best approximations to f in the classes $V_1(\alpha)$ and $V_1(\beta)$, respectively. The Tchebycheff theory applies to each so that each function $Q_\alpha - f$ and $Q_\beta - f$ has an alternant of at least 3 points. Let $Q_\alpha - f$ have m points in an alternant. If $E(\beta) \leqslant E(\alpha)$, then $Q_\alpha - Q_\beta = (Q_\alpha - f) - (Q_\beta - f)$ must have at least $m-1$ zeros (interlacing the points of an alternant for $Q_\alpha - f$.) But the maximum number of intersections for different functions in V_1 is 3. Hence, if, for some α, $Q_\alpha - f$ has 5 or more points in an alternant, then Q_α is the unique best approximant.

At the other extreme, assume that, for each α, $Q_\alpha - f$ has exactly 3 points in an alternant. It can then be shown that if $Q_\alpha(x) = e^{\alpha x}(ax+b)$ with $a \neq 0$, then $E(\alpha) > E$. The idea is to consider 2 zeros of $Q_\alpha - f$, say x_1 and x_2, which interlace the 3 points of an alternant, and to choose γ in a semineighbourhood of α. Coefficients c and d are chosen so that $Q(x) = e^{\gamma x}(cx+d)$ agrees with f at x_1 and x_2. Then Q is a better approximant to f than Q_α. Thus all best approximants are of the form $Q_\alpha(x) = be^{\alpha x}$. If there are two different such, then they must intersect in 2 points, an impossibility. Hence, again we have a unique best approximant.

For the intermediate case, assume that for some (at least one) values of α, $Q_\alpha - f$ has 4 points in an alternant, and that none has more than 4. If there is a best approximant of the form $be^{\alpha x}$, then, as above, it is the unique best approximant. Let us assume otherwise; thus any α giving a best approximation is such that $Q_\alpha - f$ has 4 points in an alternant. If there are 2 such, say α and β, let x_1 be the first point in an alternant for $Q_\alpha - f$, and let y_1 be the first point in an alternant for $Q_\beta - f$. It can be shown that the functional values are of opposite sign; i.e., $(Q_\alpha(x_1) - f(x_1))(Q_\beta(y_1) - f(y_1)) = -E^2(\alpha)$. Hence there are at most two best approximants.

4. EXAMPLES

In our first example of nonexistence, F shows bounded oscillatory behavior in contrast to the hypotheses of Theorem 1;

$$\begin{aligned} F(x) &= 2 && \text{on } [0, 2\pi], \\ &= 2 + \delta_m \sin(x/2^m) && \text{on } [2^{m+1}\pi, 2^{m+2}\pi],\ m = 0, 1,..., \end{aligned} \tag{4.1}$$

where δ_m increases steadily from 0 to 1. $F(2^m x)$ converges uniformly on $[2\pi, 4\pi]$ to $f(x) = 2 + \sin x$, a function of $C[2\pi, 4\pi]$. For this f, $E^{(+)} = 0$; but it is clear there is no polynomial P and α such that $F(\alpha x)\, P(x) = f(x)$.

In our second example, F increases steadily to ∞ at ∞, but not so quickly as the functions specified in Theorem 2;

$$F(x) = 1 + x^2. \tag{4.2}$$

For $\alpha > 0$, define $P_\alpha(x) = \alpha^{-3} + \alpha^{-2}x$. Then $F(\alpha x)\, P_\alpha(x)$ converges uniformly to $f(x) = x^3$, a function of $C[0, 1]$. Thus, again $E^{(+)} = 0$, but there is no linear polynomial P nor α such that $f(x) = P(x)\, F(\alpha x)$. The example also has relevance to Theorem 3. Here F is a rational function, but the degree of the numerator exceeds that of the denominator.

As a function which decreases steadily to 0 at ∞, we consider

$$F(x) = 1/(1 + x^2). \tag{4.3}$$

Then $\alpha^2 F(\alpha x)$ converges uniformly on $[1, 2]$ to $f(x) = 1/x^2$, a function of $C[1, 2]$. There is no polynomial P (of any degree) nor α such that $F(\alpha x)\, P(x)$ equals $f(x)$. This example also applies to Theorem 3, for which the interval $[a, b]$ has $a = 0$. Let

$$\begin{aligned} f(x) &= (x + \tfrac{7}{2})\, e^{-x/4}, && x \text{ in } [0, \tfrac{1}{2}], \\ &= f(1 - x), && x \text{ in } [\tfrac{1}{2}, 1], \end{aligned} \tag{4.4}$$

where f has a peak value at $x = \frac{1}{2}$ and equal minimum values at $x = 0$ and $x = 1$. The best approximation to f by a constant function is given by

$\frac{1}{2}[f(\frac{1}{2})-f(0)]$. Since there are 3 points in an alternant, this also gives the best approximation by a linear function. Thus, with $n=1$,

$$E(0)=\tfrac{1}{2}[f(\tfrac{1}{2})-f(0)]=2e^{-1/8}-\tfrac{7}{4}.$$

We show that $E(-\frac{1}{4})<E(0)$. Thus the value of α producing $\inf E(\gamma)$ is not 0 and does not lead to a constant function. By the calculation in Section 3, since f is invariant under transformation T, there are two different best approximants. In fact, to show $E(-\frac{1}{4})<E(0)$, it is enough to consider the function $Q(x)=(x+\frac{7}{2})\,e^{-x/4}$ on $[0,1]$;

$$\begin{aligned} Q(x)-f(x)&=0 && \text{on } [0,\tfrac{1}{2}],\\ &=\int_{1/2}^{x}[Q'(t)-f'(t)]\,dt, && x \text{ in } [\tfrac{1}{2},1]. \end{aligned}$$

Since $f'(t)=-Q'(1-t)$ and $Q'(\frac{1}{2})=0$, the above equals

$$\int_{1/2}^{x}dt\int_{1/2}^{t}[Q''(s)-Q''(1-s)]\,ds.$$

Since the third derivative of Q is positive over the given range, the integrand is positive, and

$$0\leqslant Q(x)-f(x)\leqslant Q(1)-f(1)=(9e^{-1/4}-7)/2, \quad x \text{ in } [\tfrac{1}{2},1].$$

The latter quantity is less than $E(0)$, as desired.

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